Linear Algebra II

20/03/2014, Thursday, 14:00-16:00

1 $(8+7+7=22$ pts) **Inner product spaces**

Consider the vector space $\mathbb{R}^{2\times 2}$. Let

$$\langle A, B\rangle = \mathrm{tr}(A^T B)$$

where tr denotes the sum of the diagonal elements.

(a) Show that $\langle A, B\rangle$ is an inner product.

(b) Find the distance between the matrices $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}$.

(c) Find the angle between the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

2 $(15+7=22$ pts) **Diagonalization**

(a) Find an orthogonal matrix that diagonalizes the matrix $\begin{bmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix}$.

(b) Without finding its eigenvalues, determine whether or not the matrix $\begin{bmatrix} i & -1 & 1 \\ 1 & -i & -1 \\ -1 & 1 & i \end{bmatrix}$ is unitarily diagonalizable.

3 $(15+7=22$ pts) **Singular value decomposition**

(a) Compute the singular value decomposition of the matrix $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$.

(b) Find the closest (with respect to Frobenius norm) matrix of rank 2 to M.

4 $(6+6+6+6=24$ pts) **Eigenvalues**

(a) Let A be a nonsingular matrix. Show that if λ is an eigenvalue of A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

(b) Show that the determinant of an orthogonal matrix is either -1 or 1.

(c) Show that eigenvalue of an orthogonal matrix must have modulus 1. [**Hint:** Modulus of a complex number z is defined by $\|z\| = (\bar{z}z)^{1/2}$.]

(d) Let M be a normal matrix. Show that if all eigenvalues are equal to 1 then $M = I$.

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10 pts free